

Variational Lie derivative and cohomology classes*

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Abstract

We relate cohomology defined by a system of local Lagrangian with the cohomology class of the system of local variational Lie derivative, which is in turn a local variational problem; we show that the latter cohomology class is zero, since the variational Lie derivative ‘trivializes’ cohomology classes defined by variational forms. As a consequence, conservation laws associated with symmetries of the second variational derivative of a local variational problem are globally defined.

Key words: fibered manifold, jet space, Lagrangian formalism, variational sequence, cohomology, symmetry, conservation law.

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1 Introduction and preliminaries

The geometrical formulations of the calculus of variations on fibered manifolds include a large class of theories for which the Euler–Lagrange operator is a morphism of an exact sequence [1, 8, 9, 10, 11]. The module in degree $n + 1$, consequently contains ‘equations’, *i.e.* dynamical form; the global inverse problems becomes then simple homological algebra: a given equation is an Euler–Lagrange equation if its dynamical form is the differential of a Lagrangian and this is equivalent to the ‘equation’ being closed in the complex and its cohomology class being trivial, *i.e.* Helmholtz conditions. ‘Equations’ which are only *locally variational*, *i.e.* which are closed in the complex and define a non trivial cohomology class admit a system of local Lagrangians, one for each open set in a suitable covering, which satisfy certain relations among them. We shall consider generalized symmetries, *i.e.* global projectable vector field on a jet fiber manifold which are symmetries of dynamical forms, in particular of locally variational dynamical forms. This means that symmetries of the equations are chosen and corresponding formulations of Noether theorem (II) are considered

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in order to study obstruction to globality and relative properties of associated conserved quantities.

We derive the local and global version of Noether theorems and the ‘inner’ structure of the obstruction to the existence of global conserved currents that arises is investigated. We shall explicate relevant properties of the *variational Lie derivative*, a differential operator acting on equivalence classes of variational forms in the variational sequence defined in [5], and relate the cohomology class defined by a system of local Lagrangian with the cohomology class defined by the system of local variational Lie derivative, which is in turn a local variational problem. We show that the variational Lie derivative ‘trivializes’ cohomology classes defined by variational forms. The obstruction to the existence of a global conserved current for a local variational problem is the difference of two independent cohomology classes defined by means of the variational Lie derivative. As a consequence we find that conservation laws associated with symmetries of the *second variational derivative* of a local variational problem are globally defined.

We shall consider the variational sequence [8] defined on a fibered manifold $\pi : \mathbf{Y} \rightarrow \mathbf{X}$, with $\dim \mathbf{X} = n$ and $\dim \mathbf{Y} = n + m$. For $r \geq 0$ we have the r -jet space $J_r \mathbf{Y}$ of jet prolongations of sections of the fibered manifold π . We have also the natural fiberings $\pi_s^r : J_r \mathbf{Y} \rightarrow J_s \mathbf{Y}$, $r \geq s$, and $\pi^r : J_r \mathbf{Y} \rightarrow \mathbf{X}$; among these the fiberings π_{r-1}^r are *affine bundles* which induce the natural fibered splitting [8]

$$J_r \mathbf{Y} \times_{J_{r-1} \mathbf{Y}} T^* J_{r-1} \mathbf{Y} \simeq J_r \mathbf{Y} \times_{J_{r-1} \mathbf{Y}} (T^* \mathbf{X} \oplus V^* J_{r-1} \mathbf{Y}).$$

The above splitting induces also a decomposition of the exterior differential on \mathbf{Y} in the *horizontal* and *vertical differential*, $(\pi_r^{r+1})^* \circ d = d_H + d_V$. A *projectable vector field* on \mathbf{Y} is defined to be a pair (Ξ, ξ) , where the vector field $\Xi : \mathbf{Y} \rightarrow T\mathbf{Y}$ is a fibered morphism over the vector field $\xi : \mathbf{X} \rightarrow T\mathbf{X}$. By $(j_r \Xi, \xi)$ we denote the jet prolongation of (Ξ, ξ) , and by $j_r \Xi_H$ and $j_r \Xi_V$ the horizontal and the vertical part of $j_r \Xi$, respectively.

For $q \leq s$, we consider the standard sheaves Λ_s^p of p -forms on $J_s \mathbf{Y}$, the sheaves $\mathcal{H}_{(s,q)}^p$ and \mathcal{H}_s^p of *horizontal forms*, *i.e.* of local *fibered morphisms* over π_q^s and π^s of the type $\alpha : J_s \mathbf{Y} \rightarrow \bigwedge^p T^* J_q \mathbf{Y}$ and $\beta : J_s \mathbf{Y} \rightarrow \bigwedge^p T^* \mathbf{X}$, respectively. We also have the subsheaf $\mathcal{C}_{(s,q)}^p \subset \mathcal{H}_{(s,q)}^p$ of *contact forms*, *i.e.* of sections $\alpha \in \mathcal{H}_{(s,q)}^p$ with values into $\bigwedge^p (\mathcal{C}_q^*[\mathbf{Y}])$.

According to [8], the above fibered splitting yields the *sheaf splitting* $\mathcal{H}_{(s+1,s)}^p = \bigoplus_{t=0}^p \mathcal{C}_{(s+1,s)}^{p-t} \wedge \mathcal{H}_{s+1}^t$, which restricts to the inclusion $\Lambda_s^p \subset \bigoplus_{t=0}^p \mathcal{C}_{t=0}^{p-t} \wedge \mathcal{H}_{s+1}^{t,h}$, where $\mathcal{H}_{s+1}^{p,h} := h(\Lambda_s^p)$ for $0 < p \leq n$ and the map h is defined to be the restriction to Λ_s^p of the projection of the above splitting onto the nontrivial summand with the highest value of t . Starting from this splitting one can define the sheaves of contact forms, *i.e.* forms which do not have a variational role. In fact, let us denote by $d \ker h$ the sheaf generated by the corresponding presheaf

and set then $\Theta *_r \equiv \ker h + d \ker h$. The quotient sequence

$$0 \rightarrow \mathbb{R}_{\mathbf{Y}} \rightarrow \dots \xrightarrow{\mathcal{E}_{n-1}} \Lambda_r^n / \Theta_r^n \xrightarrow{\mathcal{E}_n} \Lambda_r^{n+1} / \Theta_r^{n+1} \xrightarrow{\mathcal{E}_{n+1}} \Lambda_r^{n+2} / \Theta_r^{n+2} \xrightarrow{\mathcal{E}_{n+2}} \dots \xrightarrow{d} 0$$

defines the r -th order variational sequence associated with the fibered manifold $\mathbf{Y} \rightarrow \mathbf{X}$. It turns out that it is an exact resolution of the constant sheaf $\mathbb{R}_{\mathbf{Y}}$ over \mathbf{Y} .

The quotient sheaves (the sections of which are classes of forms modulo contact forms) in the variational sequence can be represented as sheaves \mathcal{V}_r^k of k -forms on jet spaces of higher order. In particular, currents are classes $\nu \in (\mathcal{V}_r^{n-1})_{\mathbf{Y}}$; Lagrangians are classes $\lambda \in (\mathcal{V}_r^n)_{\mathbf{Y}}$, while $\mathcal{E}_n(\lambda)$ is called a Euler–Lagrange form (being \mathcal{E}_n the Euler–Lagrange morphism); dynamical forms are classes $\eta \in (\mathcal{V}_r^{n+1})_{\mathbf{Y}}$ and $\mathcal{E}_{n+1}(\eta) := \tilde{H}_{d\eta}$ is a Helmholtz form (being \mathcal{E}_{n+1} the corresponding Helmholtz morphism).

The cohomology groups of the corresponding complex of global sections

$$0 \rightarrow \mathbb{R}_{\mathbf{Y}} \rightarrow \dots \xrightarrow{\mathcal{E}_{n-1}} (\Lambda_r^n / \Theta_r^n)_{\mathbf{Y}} \xrightarrow{\mathcal{E}_n} (\Lambda_r^{n+1} / \Theta_r^{n+1})_{\mathbf{Y}} \xrightarrow{\mathcal{E}_{n+1}} (\Lambda_r^{n+2} / \Theta_r^{n+2})_{\mathbf{Y}} \xrightarrow{\mathcal{E}_{n+2}} \dots \xrightarrow{d} 0$$

will be denoted by $H_{\text{VS}}^*(\mathbf{Y})$.

For any sheaf \mathcal{S} of Abelian groups over a topological space \mathfrak{T} , and any countable open covering of \mathfrak{T} , denoted by $\mathfrak{U} \equiv \{U_i\}_{i \in I}$, with $I \subset \mathbb{Z}$, denote the set of q -cochains with coefficients in \mathcal{S} by $C^q(\mathfrak{U}, \mathcal{S})$. Let $\sigma = (U_{i_0}, \dots, U_{i_{q+1}}) \subset \mathfrak{U}$ be a q -simplex and $f \in C^q(\mathfrak{U}, \mathcal{S})$. The *coboundary operator* $\mathfrak{d} : C^q(\mathfrak{U}, \mathcal{S}) \rightarrow C^{q+1}(\mathfrak{U}, \mathcal{S})$ is the map defined by $\mathfrak{d}f(\sigma) \equiv \sum_{i=0}^{q+1} (-1)^i r_{|\sigma|}^{\sigma_i} f(\sigma_i)$, where $\sigma_j \equiv (U_{i_0}, \dots, U_{i_{j-1}}, U_{i_{j+1}}, \dots, U_{i_{q+1}})$, for $0 \leq j \leq q+1$, r is the restriction mapping of \mathcal{S} and $|\sigma_j|$ denotes the length of σ_j . The coboundary \mathfrak{d} is a group morphism, such that $\mathfrak{d}^2 = 0$. Hence we have the cochain complex $C^0(\mathfrak{U}, \mathcal{S}) \rightarrow C^1(\mathfrak{U}, \mathcal{S}) \rightarrow C^2(\mathfrak{U}, \mathcal{S}) \rightarrow \dots$. The derived groups $H^*(\mathfrak{U}, \mathcal{S})$ of the above cochain complex are the *Čech cohomology* of the covering \mathfrak{U} with coefficients in \mathcal{S} . The *Čech cohomology* $H^*(\mathfrak{T}, \mathcal{S})$ of \mathfrak{T} with coefficients in \mathcal{S} is defined as the direct limit.

Since the variational sequence is a soft resolution of the constant sheaf $\mathbb{R}_{\mathbf{Y}}$ over \mathbf{Y} , the cohomology of the complex of global sections is naturally isomorphic to both the Čech cohomology of \mathbf{Y} with coefficients in the constant sheaf \mathbb{R} and the de Rham cohomology $H_{\text{dR}}^k \mathbf{Y}$ [8].

2 Local variational problems and cohomology

Let $\mathbf{K}_r := \text{Ker } \mathcal{E}_n$ and let $\mathcal{E}_n(\mathcal{V}_r^n)$ be the sheave of Euler–Lagrange morphisms: for a global section $\eta \in (\mathcal{V}_r^{n+1})_{\mathbf{Y}}$ we have $\eta \in (\mathcal{E}_n(\mathcal{V}_r^n))_{\mathbf{Y}}$ if and only if $\mathcal{E}_{n+1}(\eta) = 0$, which are the Helmholtz conditions of local variationality. A global inverse problem is to find necessary and sufficient conditions for such a locally variational η to be globally variational. We notice that the short exact sequence of sheaves

$$0 \rightarrow \mathbf{K}_r \rightarrow \mathcal{V}_r^n \xrightarrow{\mathcal{E}_n} \mathcal{E}_n(\mathcal{V}_r^n) \rightarrow 0$$

gives rise to the long exact sequence in Čech cohomology

$$0 \rightarrow (\mathbf{K}_r)_Y \rightarrow (\mathcal{V}_r^n)_Y \rightarrow (\mathcal{E}_n(\mathcal{V}_r^n))_Y \xrightarrow{\delta} H^1(Y, \mathbf{K}_r) \rightarrow 0.$$

Hence, every $\eta \in (\mathcal{E}_n(\mathcal{V}_r^n))_Y$ (*i.e.* locally variational) defines a cohomology class

$$\delta[\eta] \in H^1(Y, \mathbf{K}_r) \simeq H_{VS}^{n+1}(Y) \simeq H_{dR}^{n+1}(Y) \simeq H^{n+1}(Y, \mathbf{R}).$$

Analogously, let $\mathbf{T}_r := \text{Ker } d_H$; the short exact sequence of sheaves

$$0 \rightarrow \mathbf{T}_r \rightarrow \mathcal{V}_r^{n-1} \xrightarrow{d_H} d_H(\mathcal{V}_r^{n-1}) \rightarrow 0$$

gives rise to the long exact sequence in Čech cohomology

$$0 \rightarrow \Gamma(Y, \mathbf{T}_r) \rightarrow \Gamma(Y, \mathcal{V}_r^{n-1}) \rightarrow \Gamma(Y, d_H(\mathcal{V}_r^{n-1})) \xrightarrow{\delta'} H^1(Y, \mathbf{T}_r) \rightarrow 0.$$

Hence, every $\mu \in (d_H(\mathcal{V}_r^{n-1}))_Y$ (*i.e.* variationally trivial) defines a cohomology class

$$\delta'[\mu] \in H^1(Y, \mathbf{T}_r) \simeq H_{VS}^n(Y) \simeq H_{dR}^n(Y) \simeq H^n(Y, \mathbf{R}).$$

The solution to global inverse problem is now simple and elegant: η is globally variational if and only if $\delta[\eta] = 0$, because only then there exists a global section $\lambda \in (\mathcal{V}_r^n)_Y$ with $\eta = \mathcal{E}_n(\lambda)$. If instead $\delta[\eta] \neq 0$ then $\eta = \mathcal{E}_n(\lambda)$ can be solved only locally, *i.e.* for any countable good covering of Y , $\mathfrak{U} \equiv \{U_i\}_{i \in I}$, $I \subset \mathbb{Z}$, there exist local Lagrangians λ_i over each subset $U_i \subset Y$ such that $\eta_i = \mathcal{E}_n(\lambda_i)$. The local Lagrangians satisfy $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i \cap U_j}) = 0$ and conversely any system of local sections with this property gives rise to an Euler–Lagrange form $\eta \in (\mathcal{E}_n(\mathcal{V}_r^n))_Y$ with cohomology class $\delta[\eta] \in H^1(Y, \mathbf{K}_r)$.

A system of local sections λ_i of $(\mathcal{V}_r^n)_{U_i}$ for an arbitrary covering $\{U_i\}_{i \in \mathbb{Z}}$ in Y such that $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i \cap U_j}) = 0$, is what we call a *local variational problem*; two local variational problems are *equivalent* if and only if they give rise to the same Euler–Lagrange form. The covering \mathfrak{U} of Y together with the local Lagrangians λ_i is called a *presentation* of the local variational problem [6]. Note that every cohomology class in $H_{dR}^{n+1}(Y) \simeq H^{n+1}(Y, \mathbf{R})$ gives rise to local variational problems. Non trivial $H^{n+1}(Y, \mathbf{R})$ can appear *e.g.* when dealing with symmetry breaking, Y will then be fibred (over X) by homogeneous spaces. Two equivalent systems of local Lagrangians already defined with respect to the same covering can differ by an arbitrary 0-cocycle of variationally trivial Lagrangians, *i.e.* an arbitrary collections of local sections (over the $U_i \subset Y$) of \mathbf{K}_r . In consequence, on a give open set, the local Lagrangian from one system will have in general infinitesimal symmetries different from those of the local Lagrangian from the other.

For any countable open covering of Y , $\lambda = \{\lambda_i\}_{i \in I}$ is then a 0-cochain of Lagrangians in Čech cohomology with values in the sheaf \mathcal{V}_r^n , *i.e.* $\lambda \in C^0(\mathfrak{U}, \mathcal{V}_r^n)$. By an abuse of notation we shall denote by η_λ the 0-cochain formed by the restrictions $\eta_i = \mathcal{E}_n(\lambda_i)$. Let $\mathfrak{d}\lambda = \{\lambda_{ij}\} = (\lambda_i - \lambda_j)|_{U_i \cap U_j}$. Of course, $\mathfrak{d}\lambda = 0$ if and only if λ is globally defined on Y ; analogously, if $\eta \in C^0(\mathfrak{U}, \mathcal{V}_r^{n+1})$, then $\mathfrak{d}\eta = 0$ if and only if η is global. Let $\lambda \in C^0(\mathfrak{U}, \mathcal{V}_r^n)$ and let $\eta_\lambda \equiv \mathcal{E}_n(\lambda) \in C^0(\mathfrak{U}, \mathcal{V}_r^{n+1})$ be as above. We stress that $\mathfrak{d}\lambda = 0$ implies $\mathfrak{d}\eta_\lambda = 0$, while by \mathbb{R} -linearity we have $\mathfrak{d}\eta_\lambda = \eta_{\mathfrak{d}\lambda} = 0$ *i.e.* $\mathfrak{d}\lambda \in C^1(\mathfrak{U}, \mathbf{K}_r)$ [2].

2.1 Variational Lie derivative and cohomology classes

In order to formulate Noether theorems linking symmetries of the local variational problem to conserved quantities, in [6] we tackled the question what the most natural choice for *symmetries of the local variational problem* might be. We shall use extensively the concept of a *variational Lie derivative* operator $\mathcal{L}_{j_r, \Xi}$, defined for any projectable vector field (Ξ, ξ) , which was inspired by the fact that the standard Lie derivative of forms with respect to a projectable vector field preserves the contact structure induced by the affine bundles π_{r-1}^r (with $r \geq 1$) [7]. The variational Lie derivative is a local differential operator by which symmetries of Lagrangian and dynamical forms, and corresponding Noether theorems, can be characterized; thus, we shall refer to Noether theorems as formulated in terms of variational Lie derivatives of equivalence classes in the variational sequence [5].

Let η_λ be the Euler–Lagrange morphism of a local variational problem and let $\mathcal{L}_{j_r, \Xi} \eta_\lambda = 0$. Locally we have $\Xi_V \lrcorner \eta_\lambda = \Xi_V \lrcorner \mathcal{E}_n(\lambda_i)$. The first Noether theorem implies that $0 = \Xi_V \lrcorner \eta_\lambda + d_H(\epsilon_i(\lambda_i, \Xi) - \beta(\lambda_i, \Xi))$, where $\epsilon_i := \epsilon_i(\lambda_i, \Xi) = j_r \Xi_V \lrcorner p_{d_V \lambda_i} + \xi \lrcorner \lambda_i$ is the usual *canonical* Noether current. Along the solutions of Euler–Lagrange equations we thus get a *local conservation law*. We notice that in [3] local conserved currents are derived by using Lepagian equivalent of local systems of Lagrangians. The conserved current is $\epsilon(\lambda_i, \Xi) - \beta(\lambda_i, \Xi)$ which is a *local* object; in fact, the Noether current $\epsilon(\lambda_i, \Xi)$ is conserved if and only if Ξ is also a symmetry of λ_i . A local variational problem is a global object in the sense that it has a global Euler–Lagrange morphism defining a topological invariant. Consequently, there is also a precise relation between our local conservation laws [6].

In fact, let η_λ be the Euler–Lagrange morphism of a local variational problem and λ_i the system of local Lagrangians of an arbitrary given presentation. The contraction $\Xi_V \lrcorner \eta_\lambda$ defines a cohomology class, since $0 = \mathcal{L}_{j_r, \Xi} \eta = \mathcal{E}_n(\Xi_V \lrcorner \eta_\lambda)$. Then the local currents satisfy $d_H(\epsilon(\lambda_i, \Xi) - \beta(\lambda_i, \Xi) - \epsilon(\lambda_j, \Xi) + \beta(\lambda_j, \Xi)) = 0$. Thus we have that the local currents are the restrictions of a global conserved current if and only if the cohomology class $[\Xi_V \lrcorner \eta_\lambda] \in H_{dR}^n(\mathbf{Y})$ vanishes. It is noteworthy that also when the cohomology class $[\mathcal{E}_n(\lambda)]$ is trivial, the cohomology class $[\Xi_V \lrcorner \mathcal{E}_n(\lambda)]$ may be instead non trivial [6].

Remark 1 If Ξ is a symmetry of all local Lagrangians λ_i of a given presentation of the local variational problem, the Noether currents are conserved and form a system of local potentials of the cohomology class $[\Xi_V \lrcorner \eta_\lambda] \in H_{dR}^n(\mathbf{Y})$. In general, we have $d_H(\epsilon(\lambda_i, \Xi) - \epsilon(\lambda_j, \Xi)) = \mathcal{L}_{j_r, \Xi} \lambda_i - \mathcal{L}_{j_r, \Xi} \lambda_j \neq 0$, thus neither the $\mathcal{L}_{j_r, \Xi} \lambda_i$ nor the $d_H(\epsilon(\lambda_i, \Xi))$ are generally the restrictions of global closed n -forms. \square

There are in fact two rather independent obstructions, one coming from the Lie derivative of the local Lagrangians being not necessarily zero, the other from the system of local Noether currents. In the cohomological obstruction to the existence of a global Lagrangian for a local variational problem, it is then of great interest to study how the variational Lie derivative affects cohomology

classes.

Proposition 1 *The variational Lie derivative transforms non trivial cohomology classes to trivial cohomology classes associated with the variational Lie derivative of local presentations, i.e. the variational Lie derivative ‘trivializes’ cohomology classes in the variational sequence.*

PROOF. In fact, by linearity we have

$$\eta_{\mathcal{L}_{\Xi}\lambda_i} = \mathcal{E}_n(\Xi \lrcorner \eta_\lambda) + \mathcal{E}_n(d_H \epsilon_i) = \mathcal{E}_n(\Xi \lrcorner \eta_\lambda) = \mathcal{L}_{\Xi}\eta_\lambda.$$

Since $\mathcal{L}_{\Xi}\eta_\lambda = \mathcal{E}_n(\Xi \lrcorner \eta_\lambda)$ we have that $\delta(\mathcal{L}_{\Xi}\eta_\lambda) = \delta(\eta_{\mathcal{L}_{\Xi}\lambda_i}) = 0$ although $\delta(\eta_\lambda) \neq 0$.

This result, by the way, holds true at any degree k in the variational sequence and independently from the fact that Ξ be a generalized symmetry or not. In particular, from this we deduce the following.

Corollary 1 *Euler–Lagrange equations of the local problem defined by $\mathcal{L}_{\Xi}\lambda_i$ are equal to Euler–Lagrange equations of the global problem defined by $\Xi \lrcorner \eta_\lambda$.*

In other words, the local problem defined by the local presentation $\mathcal{L}_{\Xi}\lambda_i$ is variationally equivalent to a global one.

The same can be stated for local variationally trivial Lagrangians. Suppose we have a global variationally trivial Lagrangian μ , i.e. such that $\mathcal{E}_n(\mu) = 0$, this means that we have a 0-cocycle of currents ν_i such that $\mu = d_H \nu_i$ and $\mathfrak{d}\mu_\nu = 0$ but we suppose $\delta'(\mu_\nu) \neq 0$. We can consider the Lie derivative $\mathcal{L}_{\Xi}\nu_i$ and the corresponding $\mu_{\mathcal{L}_{\Xi}\nu_i}$.

Corollary 2 *Divergence equations (conditions for a Lagrangian to be locally variationally trivial) of the local problem defined by $\mathcal{L}_{\Xi}\nu_i$ are equal to divergence equations of the global problem defined by $\Xi \lrcorner \lambda_\nu$.*

PROOF. We have

$$\mu_{\mathcal{L}_{\Xi}\nu_i} = d_H(\Xi \lrcorner \mu_\nu) = \mathcal{L}_{\Xi}\mu_\nu,$$

so that $\delta'(\mathcal{L}_{\Xi}\mu_\nu) = \delta'(\mu_{\mathcal{L}_{\Xi}\nu_i}) = 0$, although $\delta'(\mu_\nu) \neq 0$.

When Ξ is a generalized symmetry we have $\mathcal{L}_{\Xi}\eta_\lambda = 0$ then, in particular, we have $\delta(\mathcal{L}_{\Xi}\eta_\lambda) \equiv 0$. Furthermore, if Ξ is a generalized symmetry then $\mathcal{E}_n(\Xi \lrcorner \eta_\lambda) = 0$ then we have a 0-cocycle ν_i as above, defined by $\mu_\nu = \Xi \lrcorner \eta_\lambda := d_H(\nu_i)$. We see that a way to find a system of global currents associated with a system of local currents is to take the Lie derivatives of the local system, for which we saw that we would have $\delta'(\mathcal{L}_{\Xi}\mu_\nu) = \delta'(\mathcal{L}_{\Xi}(\Xi \lrcorner \eta_\lambda)) = \delta'(\mathcal{L}_{\Xi}(d_H(\nu_i))) = \delta'(d_H(\mathcal{L}_{\Xi}(\nu_i))) = 0$.

A natural question is now if there exist a way to find under which conditions such a variational Lie derivative of local currents is a system of *conserved* currents. The answer to such a question involves Jacobi equations for the local system λ_i .

Proposition 2 *Let Ξ be a symmetry of the Euler–Lagrange form η_λ . If the second variational derivative is vanishing, then we have the conservation law $d_H \mathcal{L}_\Xi(\nu_i + \epsilon_i) = 0$, where $\mathcal{L}_\Xi(\nu_i + \epsilon_i)$ is a local representative of a global conserved current.*

PROOF. In fact, let us apply twice the variational Lie derivative; since we are supposing that Ξ is a *generalized symmetry*, we have

$$\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = \mathcal{L}_\Xi(\Xi \lrcorner \eta_\lambda) + \mathcal{L}_\Xi(d_H(\epsilon_i)) = \mathcal{L}_\Xi(d_H(\nu_i)) + \mathcal{L}_\Xi(d_H(\epsilon_i)) = d_H \mathcal{L}_\Xi(\nu_i + \epsilon_i).$$

Thus the statement is an immediate consequence of the condition $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i := \delta^2 \lambda_i = 0$. Notice that $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$ means that the generalized symmetry Ξ is required to be a generalized symmetry *and* also a symmetry of the local variational problem $\mathcal{L}_\Xi \lambda_i$.

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